# Question 3

Suggested time: 20 min

Consider the Poisson model for a random variate $y$ with parameter $\lambda$, given as

$$\begin{aligned} p(y|\lambda) &= \frac{\lambda^y exp(-\lambda)}{y!}, \quad \text{with} \\ E(y|\lambda) &= V(y|\lambda) = \lambda, \quad \lambda > 0, y \in \{0,1,2,3,....\} \end{aligned} \tag{1}$$

**Part (a)**
Now consider a sample of $n$ observations from this distribution, with each observation generically labeled $y_i, i = 1 \ldots n$. Write down the joint distribution for the sample data (in *un*-logged form). Call it $p(\mathbf{y}|\lambda)$.

**Part (b)**
Suppose you stipulate a *gamma* prior density for $\lambda$ with shape parameter $a$ and inverse scale ("rate") parameter $b$, given as

$$\begin{aligned} p(\lambda) &= g(a,b) = \frac{b^a}{\Gamma(a)} \lambda^{(a-1)} exp(-b\lambda), \quad \text{with} \\ E(\lambda) &= \frac{a}{b}, \; V(\lambda) = \frac{a}{b^2}, \quad \lambda, a, b > 0, \end{aligned} \tag{2}$$

Show that the posterior distribution of $\lambda$, given your collected data from the Poisson, is also a gamma. Show the form of the posterior shape and rate parameters (you can call them $a^*$ and $b^*$).

**Part (c)**
Show that the posterior expectation can be written as a weighted average of the prior expectation and the sample mean. What happens to this posterior expectation as $n \to \infty$?

**Part (d)**
Suppose you are opening a small restaurant In Blacksburg. Before you start your business, you expect 20 guests / day with a variance of 10, which can be modeled as a gamma prior with shape 40 and rate 2. After 30 days of running your business, you count a total of 824 guests. You plot the daily counts, and they look exactly like a Poisson distribution.

How many guest *per day* would you expect for the following month?