AAEC/ECON 5126 MIDTERM EXAM: SOLUTIONS

SPRING 2014 / INSTRUCTOR: KLAUS MOELTNER

This exam is open-book, open-notes, but please work strictly on your own. Please make sure your name is on every sheet you're handing in. You have 75 minutes to complete this exam. You can collect a maximum of 30 points. Each question is scored as indicated below. Vectors are given in lower-case boldface. Matrices are written in upper-case boldface.

1. Question I (10 points)

Consider the CLRM for a full sample of n observations: $\mathbf{y} = \beta_0\mathbf{i} + \beta_1\mathbf{d} + \beta_2\mathbf{x} + \boldsymbol{\epsilon}$, where all the usual assumptions hold, $\mathbf{i}$ is an n by 1 vector of "1"s, $\mathbf{d}$ is an indicator variable that takes the value of "1" for person i if that person belongs to a specific category (e.g. "female", or "hispanic", etc.), and a value of "0" otherwise, and $\mathbf{x}_i$ is a continuous explanatory variable. Assume that n_1 of the n individuals belong to the indexed category (with $0 < n_1 < n$).

(a) *(2 pts.)* For a single observation, show the form of $E(y_i|d_i = 0, x_i)$, and $E(y_i|d_i = 1, x_i)$. Is the marginal effect of x_i on $E(y_i|x_i)$ the same for both groups or not? How about the intercept?

(b) *(2 pts.)* Let $\mathbf{z} = \begin{bmatrix}\mathbf{i} & \mathbf{d}\end{bmatrix}$ and $\boldsymbol{\beta}_z = \begin{bmatrix}\beta_0 & \beta_1\end{bmatrix}'$. Show the explicit form of $\mathbf{z}'\boldsymbol{\epsilon}$.*(Hint: This should be a 2 by 1 vector with two summation terms. The second summation term has a condition for the elements over which the summation is taken. You should state this condition underneath the summation sign.)*

(c) *(2 pts.)* Let b_2 be the OLS estimator for β_2. Write its solution in partitioned regression form, and show the explicit form of the residual maker matrix $\mathbf{M}_1$ in terms of $\mathbf{z}$.

(d) *(4 pts.)* Show that b_2 is an unbiased estimator for β_2 (conditional on $\mathbf{x}, \mathbf{d}$).

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Solutions:

(a)

$$E\left(y_i|d_i=0,x_i\right)=\beta_0+\beta_2*x_i$$
$$E\left(y_i|d_i=1,x_i\right)=\left(\beta_0+\beta_1\right)+\beta_2*x_i$$

The slopes (marginal effect of x_i) are the same for both groups, but the intercepts differ. That's why indicator variables like d_i are often called "intercept shifters."

(b)

$$\mathbf{z}'\boldsymbol{\epsilon}=\begin{bmatrix}\mathbf{i}'\boldsymbol{\epsilon}\\ \mathbf{d}'\boldsymbol{\epsilon}\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{n}\epsilon_i\\ \sum_{i|d_i=1}\epsilon_i\end{bmatrix}$$

The second summation is only over individuals for whom $d_i=1$.

(c)

$$b_2=\left(\mathbf{x}'\mathbf{M}_1\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_1\mathbf{y}$$
$$\mathbf{M}_1=\mathbf{I}-\mathbf{z}\left(\mathbf{z}'\mathbf{z}\right)^{-1}\mathbf{z}'$$

where $\mathbf{I}$ is an n by n identity matrix.

(d) Plugging in the full regression model for $\mathbf{y}$ in the partitioned regression form of b_2 and taking expectations yields:

$$\begin{aligned}&E\left(b_2|\mathbf{x},\mathbf{d}\right)=\\&E\left(\left(\mathbf{x}'\mathbf{M}_1\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_1\mathbf{z}\boldsymbol{\beta}_z\right)+\\&E\left(\left(\mathbf{x}'\mathbf{M}_1\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_1\mathbf{x}\beta_2\right)+\\&E\left(\left(\mathbf{x}'\mathbf{M}_1\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_1\boldsymbol{\epsilon}\right)=\beta_2\end{aligned}$$

since $\mathbf{M}_1\mathbf{z}=\mathbf{0}$ and $E\left(\mathbf{z}'\boldsymbol{\epsilon}\right)=E\left(\mathbf{x}'\boldsymbol{\epsilon}\right)=0$ (and therefore $E\left(\mathbf{x}'\mathbf{M}_1\boldsymbol{\epsilon}\right)=0$).

2. Question II (10 points)

(Continuation of Question I): Now assume the analyst ignores the binary indicator variable $\mathbf{d}$, and instead estimates the following (incorrect) model: $\mathbf{y} = \mathbf{i}\beta + \mathbf{x}\beta_2 + \boldsymbol{\nu}$.

(a) *(2 pts.)* Show the explicit form of $\boldsymbol{\nu}$.

(b) *(2 pts.)* Let $\hat{b}_2$ be the OLS estimator for β_2 for this model. Write its solution in partitioned regression form, and show the explicit form of the residual maker matrix $\mathbf{M}_1$ in this case.

(c) *(4 pts.)* Show that $\hat{b}_2$ is a *biased* estimator for β_2 (conditional on $\mathbf{x}$).

(d) *(2 pts.)* Show that - trivially - the bias vanishes if $d_i = 0, \forall i$, or $d_i = 1, \forall i$ (i.e. if $n_1 = 0$ or $n_1 = n$).

Solutions:

(a)

$$\boldsymbol{\nu} = \beta_1 \mathbf{d} + \boldsymbol{\epsilon}$$

(b)

$$\hat{b}_2 = \left(\mathbf{x}'\mathbf{M}_1\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_1\mathbf{y}$$
$$\mathbf{M}_1 = \mathbf{I} - \mathbf{i}\left(\mathbf{i}'\mathbf{i}\right)^{-1}\mathbf{i}' = \mathbf{M}_0,$$

where $\mathbf{M}_0$ is the deviation-from-mean matrix.

(c) Plugging in the full (TRUE!) regression model for $\mathbf{y}$ in the partitioned regression form of $\hat{b}_2$ and taking expectations yields:

$$\begin{aligned}
&E\left(\hat{b}_2|\mathbf{x},\mathbf{d}\right) = \\
&E\left(\left(\mathbf{x}'\mathbf{M}_0\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_0\mathbf{i}\beta_0\right) + \\
&E\left(\left(\mathbf{x}'\mathbf{M}_0\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_0\mathbf{x}\beta_2\right) + \\
&E\left(\left(\mathbf{x}'\mathbf{M}_1\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_0\mathbf{d}\beta_1\right) + \\
&E\left(\left(\mathbf{x}'\mathbf{M}_1\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_0\boldsymbol{\epsilon}\right) = \\
&\beta_2 + \left(\mathbf{x}'\mathbf{M}_1\mathbf{x}\right)^{-1}\mathbf{x}'\mathbf{M}_0\mathbf{d}\beta_1 \neq \beta_2
\end{aligned}$$

So while $\mathbf{M}_0\mathbf{i} = 0$ and $E\left(\mathbf{x}'\mathbf{M}_0\boldsymbol{\epsilon}\right) = 0$, the third expectation term does not go to zero. More explicitly:

$$\mathbf{x}'\mathbf{M}_0\mathbf{d} = \sum_{i=1}^{n}\left(x_i * \left(d_i - \bar{d}\right)\right) = \sum_{i=1}^{n}\left(x_i * d_i\right) - \bar{d}\sum_{i=1}^{n} x_i$$

where $\bar{d} = n_1$. So this term only goes to zero if $d_i = 0\,\forall i$ or $d_i = 1\,\forall i$. Note that in the former case $\bar{d} = 0$, and in the latter case $\bar{d} = 1$.

3. Question III (10 points)

Let $\hat{\theta}$ be an asymptotically normal, consistent estimator for θ, with $\theta > 0$.

(a) *(2 pts.)* Find a consistent estimator for $\gamma = log\,(\theta)$, and state your reasoning, where "log" denotes the natural logarithm. Call the estimator $\hat{\gamma}$.

(b) *(2 pts.)* Let $V_{\hat{\theta}}$ be the asymptotic variance of $\hat{\theta}$. Show the general form of the asymptotic variance of $\hat{\gamma}$ (call it $V_{\hat{\gamma}}$).

(c) *(2 pts.)* Assume you collect a sample of data, and estimate $\hat{\theta} = 4$, with $s.e.\left(\hat{\theta}\right) = 2$, where "s.e." denotes the asymptotic standard error. Compute $\hat{\gamma}$ and its asymptotic standard error.

(d) *(2 pts.)* Using the actual numbers form above, construct a z-test for $H_0 : \theta = 1$. State your decision (use $\alpha = 0.05$).

(e) *(2 pts.)* Now argue that the same null could be equivalently expressed as $H_0 : \gamma = log\,(\theta) = 0$ and perform the corresponding test. Compare your result to the previous test - what do you conclude regarding logically equivalent (asymptotic) hypothesis tests based on nonlinear transformations of parameters?

Solutions:

(a) A good candidate might be $\hat{\gamma} = log\left(\hat{\theta}\right)$. By the Slutsky theorem we have:

$$\text{plim}\left(\hat{\gamma}\right) = \text{plim}\left(log\left(\hat{\theta}\right)\right) = log\left(\text{plim}\left(\hat{\theta}\right)\right) = log\left(\theta\right) = \gamma$$

(b) Using the Delta method we have:

$$V_{\hat{\gamma}} = \left(\frac{\partial log\left(\hat{\theta}\right)}{\partial\theta}\right)^2 V_{\hat{\theta}} = \frac{1}{\hat{\theta}^2} V_{\hat{\theta}}$$

(c)

$$\hat{\gamma} = log\left(4\right) \approx 1.39$$

$$s.e\left(\hat{\gamma}\right) = \sqrt{V_{\hat{\gamma}}} = \sqrt{\frac{1}{4}} = \tfrac{1}{2}$$

(d)

$$z = \frac{\hat{\theta} - 1}{s.e.\left(\hat{\theta}\right)} = \frac{3}{2} = 1.5$$

Fail to reject the null (the critical z-value is 1.96).

(e)

$$z = \frac{\hat{\gamma}}{s.e.\left(\hat{\gamma}\right)} = \frac{1.39}{0.5} = 2.78$$

Strongly reject the null. This shows that logically equivalent (asymptotic) hypothesis tests based on nonlinear transformations can nonetheless lead to different decision rules.